Biot's Variational Principle in Heat Conduction†

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Summary

Biot's variational principle is applied to a number of different one-dimensional heat conduction problems. These problems show the applicability of the variational principle to problems involving prescribed heat flux boundary conditions and to those with temperature-dependent material properties.

A method is introduced for including boundary conditions when these are expressed as prescribed heat fluxes. The idea behind this is overall energy balance within the body, which is a constraint condition to be satisfied by the time histories of the generalized coordinates.

The variational principle is then applied to the well-known problem of constant surface heat flux in order to present the technique and provide a basis for the remaining sections. The equivalence of the result obtained in applying the variational principle for a prescribed surface temperature history to that obtained for a prescribed heat flux is also pointed out. Radiation cooling due to fourth power radiation from semi-infinite solids and finite slabs together with radiation according to Newton's law of cooling is then treated. Finally, the introduction of temperature-dependent material properties is discussed and the determination of the temperature distribution in a semi-infinite solid with variable properties is investigated.

Introduction

APPROXIMATE ANALYTICAL METHODS of solution to heat conduction problems have received considerable attention in the last few years. This attention has been due to the necessity of including nonlinear boundary conditions and material property variations into the analysis of heat conduction problems. Since classical methods of heat conduction analysis[1] can not account for these effects, approximate analytical methods which can include these two effects in a systematic manner are necessary. Two important approximate methods that have been introduced with this need in mind are due to Goodman[17–20] and Biot.[2–9]

Goodman's method—the heat balance integral technique—has been applied to many problems of nonlinear heat inputs[17–21] and to problems involving temperature-dependent material properties.[20] This latter case will be discussed subsequently.

The variational principle introduced by Biot, however, and the problems he has investigated have been confined in general to problems involving prescribed surface temperature variations. As a consequence the principle has not been exploited as much as possible; discussion of some of these problems is also found in Refs. 10 and 11.

The problem of the determination of the temperature distribution in a flange-web combination (angle section) was treated by Biot[5] and was further discussed in Refs. 12 and 13. An extension of the variational method for this problem to two dimensions was given by Levinson.[14] Citron[15, 16] in addition, has discussed Biot's principle with reference to ablation problems. In Citron's work the variational principle was expressed in terms of a functional expression involving the heat flux and the temperature gradient. The variation was carried out with respect to the heat flux with the temperature gradient held fixed. A more recent application of the principle to ablation problems was given in Ref. 32.

With the exception of this application, and that by Levinson for the angle section, the above references present the same results and examples treated by Biot which involve the heating of bodies with prescribed surface temperature boundary conditions.

The important practical case where the heat flux is prescribed at the surface has not been treated before by the variational method. For this reason, the present paper will investigate the solution of a number of one-dimensional problems with prescribed heat flux boundary conditions.

The variational principle is first reviewed and a method for accounting for a prescribed heat flux boundary condition is introduced. After the review and discussion of the variational principle, the principle is applied to problems of constant heat flux at the surface of a semi-infinite solid or finite slab. This simple case is presented to outline the technique of the variational principle in the case where a boundary condition on the heat flux is prescribed. The equivalence of the heat flux constraint condition to that of prescribed surface temperature is also discussed for a particular profile. In addition, the separation of the problem into two phases for the heating of finite slabs is reviewed.

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The introduction of nonlinear heat flux boundary conditions is then presented. In particular, the analysis is given for black body radiation from the surface of a semi-infinite solid and from one surface of a finite slab. The extension of the variational principle to problems with temperature-dependent thermal properties is applied to the solution of a problem with a fixed surface temperature. The results for all examples are presented in graphical form and a comparison is made with other methods of solutions.

Variational Principle

Although the variational principle for heat transfer introduced by Biot[5] was based on the theory of irreversible thermodynamics,[3, 4, 7] the mathematical formulation alone will be reviewed in this section.

The variational principle is stated in terms of a heat flow vector field $\mathbf{H}$, whose time rate of change $\dot{\mathbf{H}}$ is the heat flux across an area normal to $\dot{\mathbf{H}}$. Since energy must be conserved in the heat transfer process, it is necessary that

$$\int \mathbf{H}\cdot\mathbf{n}\, dS = -\int c\theta\, dv$$

or

$$-c\theta = \operatorname{div} \mathbf{H} \tag{1}$$

where θ is the temperature field above an equilibrium temperature, c is the specific heat per unit volume and $\mathbf{H}$ is the heat flow field.

The variational principle is stated as

$$\delta V + \delta D = -\int \theta\, \delta\mathbf{H}\cdot\mathbf{n}\, dS \tag{2}$$

where

$$\begin{aligned} V &= \int (1/2)c\theta^2\, dv \\ \delta D &= \int (1/k)(\dot{\mathbf{H}})\cdot\delta\mathbf{H}\, dv \end{aligned} \tag{3}$$

The function V is the thermal potential and is related to the thermal energy of the system while the dissipation function D is equal to the production of entropy in the system.

For arbitrary variations in the heat flow field, the variational principle is equivalent to the heat conduction equation

$$\nabla\cdot(k\nabla\theta) = c(\partial\theta/\partial t) \tag{4}$$

The variation in the thermal potential V yields

$$\delta V = \int c\theta\, \delta\theta\, dv \tag{5}$$

which, together with energy conservation, Eq. (1), becomes

$$\delta V = -\int \theta\delta\mathbf{H}\cdot\mathbf{n}\, dS + \int \delta\mathbf{H}\cdot\nabla\theta\, dv \tag{6}$$

The variational principle thus leads to the following equation:

$$\int [\nabla\theta + (1/k)\dot{\mathbf{H}}]\cdot\delta\mathbf{H}\, dv = 0 \tag{7}$$

For arbitrary variations $\delta\mathbf{H}$, this reduces to

$$k\nabla\theta + \dot{\mathbf{H}} = 0 \tag{8}$$

Therefore, the variational principle given by Eq. (2) which is to be satisfied for arbitrary variations in the heat flow field leads to Eq. (8). This result is Fourier's law of conduction[1] relating the heat flux $\dot{\mathbf{H}}$ to the temperature gradient. Eq. (8) with Eq. (1) yields the heat conduction equation, Eq. (4), and the equivalence of the variational principle to the heat conduction equation is established.

In any approximate solution employing the variational principle, Eq. (2), Fourier's law of conduction is approximated and not conservation of energy. Thus the fundamental requirement of energy conservation is enforced, or what is equivalent, any variations in the temperature field are constrained by variations in the heat-flow field.

While the above discussion has applied to isotropic media with constant thermal properties, the generalization to anisotropic materials is straight-forward.[5] The application to materials with temperature-dependent properties requires a modification in the definition of the thermal potential V which was also discussed by Biot.[5]

The advantage of the present variational principle is that it can be stated in terms of equations of the Lagrangian type for generalized coordinates defining the heat flow field. The heat flow vector field can be considered as a function of n generalized coordinates $q_n(t)$ in the form:

$$\mathbf{H} = \mathbf{H}(q_n;\, x,y,z,t) \tag{9}$$

where the generalized coordinates specify the configuration of the heat flow field as a function of time. The variational principle can then be written in terms of the generalized coordinates in the form[5]

$$\partial V/\partial q_i + \partial D/\partial \dot{q}_i = Q_i \tag{10}$$

where

$$D = \int (1/2\, k)\, \dot{H}^2\, dv$$

$$Q_i = \int \theta(\partial H_n/\partial q_i)\, dS$$

The Lagrangian form of the equations for the generalized coordinates, Eqs. (10), was established by Biot[3, 4, 7] on the basis of linear irreversible thermodynamics. However, since the formalism of the variational method leads to the standard form of the heat conduction equation, which is known to hold for large temperature variations, the variational principle has wider application than the thermodynamical arguments would seem to suggest. That is, the range of validity of the variational formulation is equivalent to the range of application of the heat conduction equation. This is independent of any assumptions that have been made in the thermodynamical formulations of the Lagrangian equations. The variational principle can also be extended to the general case of anisotropy and temperature-dependent properties and the general case can be formulated in Lagrangian terms.[5]

In one-dimensional cases the temperature field and the heat flow field are simply related; this simplicity

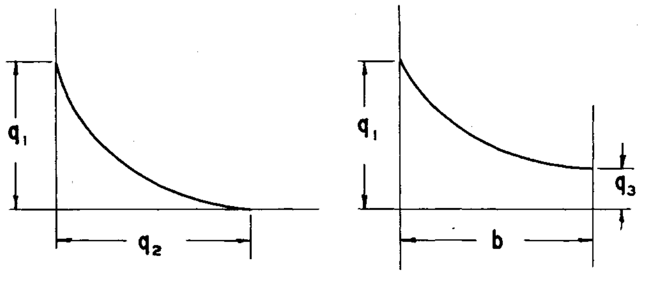

FIG. 1. Temperature profiles in terms of generalized coordinates; heat input.

is absent in two and three-dimensional cases since then divergence-free heat flow fields may be added arbitrarily [see Eq. (1)] without affecting the temperature field. The present paper will be restricted to the one-dimensional case, and here the procedure is a direct one: the temperature field is expressed in terms of generalized coordinates and the heat flow field is determined by an integration. This leads directly to equations of the Lagrangian type, Eqs. (10).

The assumed temperature profile which is expressed in terms of the generalized coordinates will generally be assumed as a polynomial, although different profiles will be discussed in the next section. The generalized coordinates for typical temperature distributions are indicated in Fig. 1, where the coordinates q_1 and q_3 are the surface temperatures and q_2 is the penetration depth. When q_2 is equal to the thickness of the slab, the inner surface temperature begins to rise. The time required for the penetration depth to equal the slab thickness is called the penetration time. Until the penetration time is attained the inner surface temperature is constant.

The concept of penetration depth has also been employed in Goodman's heat balance integral method,[17] and by Tramposch,[22] in a formulation of an approximate solution for the aerodynamic heating of a web-flange structure.

Therefore, for a finite body the formulation of the problem is naturally divided into two phases, with the time of separation corresponding to the penetration time.

The above principle and techniques have been employed by Biot in a number of examples[4, 5, 9] which have involved boundary conditions on the temperatures. Since these boundary conditions are not as frequent in physical situations as boundary conditions on the heat flux, this latter condition is of greater importance. However, since the variational principle does not introduce any boundary conditions, for example, natural boundary conditions obtained from conventional variational principles, a method for including the flux boundary condition is necessary.

The method to be employed here is to consider $(n - 1)$ of the generalized coordinates as independent coordinates to be determined from the variational principle while the nth coordinate will be determined from over-all energy balance. That is, the heat input per unit time per unit area—the prescribed flux—at the surface of the body must equal the rate of change of the heat content in the body. This expression takes the form

$$\frac{d}{dt}\left(\int c\theta\, dv\right) = \int F\, dS \tag{11}$$

where F is the prescribed heat flux at the surface of the body. Eq. (11) and Eq. (1) yield

$$-\frac{d}{dt}\int \operatorname{div} \mathbf{H}\, dv = \int \dot{\mathbf{H}}\cdot\mathbf{n}\, dS = \int F\, dS$$

or

$$\int \dot{H}_n\, dS = \int F\, dS \tag{12}$$

where $\dot{H}_n$ is the heat flux into the body, and $\mathbf{n}$ is the normal vector to the surface, the direction into the body taken as positive. For the one-dimensional case, Eq. (12) takes the form

$$\dot{H}_n = F \tag{13}$$

Eq. (13) is the flux condition and enforces over-all energy balance in the body. It is to be employed as a constraint condition with the variational principle. This equation differs from the conservation of energy equation, Eq. (1), in that the latter is a local condition between the heat flow and temperature fields, while the former is an over-all energy balance related to the heat input to the body.

If the boundary conditions are expressed in terms of surface temperatures rather than in terms of surface heat fluxes, the constraint condition is superfluous and the set of equations assumes its usual form. Furthermore, if it is possible to select a temperature field such that the corresponding heat flux field identically satisfies the prescribed flux at the surface, the constraint condition is identically satisfied, and the explicit use of the flux condition is again unnecessary.

Therefore the flux condition, Eq. (13), is used in the one-dimensional case, together with either of the two Lagrangian equations corresponding to q_1 and q_2 (q_3 for a finite slab). The generalized coordinate selected as the independent coordinate is related to the second through the expression for over-all energy conservation. In order to indicate the above method of solution, the case of constant heat flux will be treated in complete detail in the next section. This case will serve to present the details of the method and to provide a basis for the remaining sections.

Constant Heat Flux

The exact solutions for the case of constant heat flux applied to the surface of a semi-infinite solid and to a finite slab insulated at one face are known,[1] and these solutions will be used for comparison with the approximate variational solutions. This section will indicate the basic ideas in applying the variational principle and the selection of assumed temperature profiles. In addition, the approximate solutions obtained by Goodman using the heat balance integral approach will be compared with the present results.

Semi-Infinite Solid

The spatial temperature profile in the solid will be assumed here to be parabolic, while cubic and exponential profiles are treated in Refs. 29 and 31. A comparison of the results for all profiles will be made with the exact solution. The temperature fields are taken as

$$\theta = q_1[1 - (x/q_2)]^2 \tag{14}$$

$$\theta = (3Ft/cq)[1 - (x/q)]^2 \tag{15}$$

The profile given by Eq. (14) is written in terms of the surface temperature q_1 and penetration depth q_2 while Eq. (15) is written in terms of the penetration depth q, the surface temperature being expressed in terms of t, q, and the constant heat flux F. The reason for selecting this latter profile will become apparent in what follows. The parabolic profile will be discussed in detail in this section while the results for other profiles will only be stated; a complete formulation for these profiles is given in Ref. 29.

The energy equation

$$\text{div}\,\mathbf{H} = -c\theta$$

together with Eq. (15) yields the heat flow field

$$H = Ft[1 - (x/q)]^3$$

The corresponding heat flux field is

$$\dot{H} = F[1 - (x/q)]^3 + (3F\dot{q}tx/q^2)\,[1 - (x/q)]^2 \tag{16}$$

which satisfies the conditions

$$\dot{H} = 0 \quad \text{at} \quad x = q \tag{17}$$

$$\dot{H} = F \quad \text{at} \quad x = 0 \tag{18}$$

Eq. (17) corresponds to zero flux at the penetration depth and Eq. (18) corresponds to the imposed flux condition, F = constant, at the surface of the body. For the temperature field given by Eq. (15), the heat flux field $\dot{H}$ at the surface is identically equal to the applied surface heat flux and therefore satisfies the prescribed boundary condition there.

Eqs. (15) and (16) yield the following results for the derivatives of the thermal potential and dissipation function:

$$\partial V/\partial q = -(9F^2t^2/10cq^2)$$

$$\partial D/\partial \dot{q} = (F^2t/k)[(3\dot{q}t/35q) + (3/42)]$$

in addition

$$Q = 0$$

The governing differential equation

$$\partial V/\partial q + \partial D/\partial \dot{q} = Q \tag{19}$$

becomes

Table 1. Results for Constant Flux Case

		Surface Temp.	Penetration Depth	% Diff.
(I)	Parabolic			
	A	1.065	2.81	−5.6
	B	1.157	2.59	+2.0
	C	1.120	2.68	−0.7
(II)	Exponential			
	A	1.225	0.816	+8.6
	B	1.120	0.895	−0.7
	C	1.152	0.866	+2.3
(III)	Cubic			
	A	1.105	3.62	−2.0
	B	1.14	3.52	+1.0
	C	1.123	3.56	−0.5
(IV)	Goodman:			
	Parabolic	1.225	2.45	+8.6
	Cubic	1.15	3.45	+2.0
(V)	Exact	1.128	...	...

A = heat flow field satisfies flux condition
B = independent coordinate q_1, surface temperature
C = independent coordinate q_2, penetration depth

$$\alpha[(3/35)q\dot{q}t + (3/42)q^2] = (9/10)t$$

where $\alpha = k/c$ is the thermal diffusivity. The solution for the penetration depth is

$$q = 2.81(\alpha t)^{1/2}$$

The surface temperature is

$$\theta_s = (3Ft/cq) = 1.065\,(F/k)(\alpha t)^{1/2}$$

and the exact solution for the surface temperature is

$$\theta_s = 1.128\,(F/k)(\alpha t)^{1/2}$$

This particular method of solution, using Eq. (15), is limited in that it will not always be possible to select a temperature or heat flow field such that the surface flux conditions are exactly satisfied. Instead, the Lagrangian equation for one generalized coordinate must be used with the equation for overall energy balance, expressed by Eq. (13). The formulation in this case will be given for the parabolic profile, Eq. (14), the independent generalized coordinate being q_2, Fig. 1.

Eqs. (1), (10), and (14) yield

$$(\dot{\psi}\eta^2/42) + (13\psi\eta\dot{\eta}/315) = (7\psi/30) \tag{20}$$

where

$$\tau = (F/cT_0)^2(t/\alpha)$$

$$\psi = (q_1/T_0)$$

$$\eta = (q_2F/kT_0)$$

T_0 is a reference temperature and the dot symbol indicates differentiation with respect to the nondimensional time τ.

The flux condition from Eq. (13) is

$$\dot{\psi}\eta + \dot{\eta}\psi = 3 \tag{21}$$

which corresponds to constant heat input.

Eq. (20) gives the relation between the coordinates to satisfy the variational principle, while Eq. (21) is the constraint condition imposed on the coordinates as a consequence of the prescribed flux. These two equations specify the time variation of the coordinates

q_1 and q_2. If, instead of q_2 as the independent generalized coordinate q_1 had been selected, Eq. (20) would be replaced by

$$(\dot{\psi}\eta^2/63) + (\psi\eta\dot{\eta}/42) = (2\psi/15) \qquad (22)$$

The governing differential equations for this independent generalized coordinate are Eqs. (21) and (22).

The results for the surface temperature and penetration depth (in the case of the exponential profiles, the equivalent penetration depth) are summarized in Table 1. These results are listed as the coefficients of

$$q_1 = (\ )(F/k)(\alpha t)^{1/2}$$
$$q_2 = (\ )(\alpha t)^{1/2}$$

for the parabolic profile given by Eq. (14) and for the exponential and cubic profiles obtained from Ref. 29. It can easily be verified that these solutions satisfy Eqs. (20) and (21). In those cases where two generalized coordinates are present, the results for the selection of either as the independent coordinate are given. The initial conditions for the above results are zero surface temperature and penetration depth. Table 1 also indicates the results obtained by Goodman's heat balance integral method and these are shown in group IV. There is good agreement in all cases with the exact surface temperature.

The internal temperature distribution can also be determined; this is shown in Fig. 2 for Eq. (14), and an exponential temperature distribution together with the result from the exact solution. It is seen that agreement is good over the portion corresponding to large temperature gradients. Near the penetration depth the results differ by a small percent of the surface temperature value. This is a consequence of a finite penetration depth for the parabolic profile, while for the exponential profile the decay for large x is not as rapid as for that of the exact solution.

The result obtained for the penetration depth in the parabolic profile case, where q_2 is the independent coordinate, is consistent with that obtained in Ref. 9. In this reference it was found that if the surface temperature varies as t^n where $n \geq 0$, the penetration depth

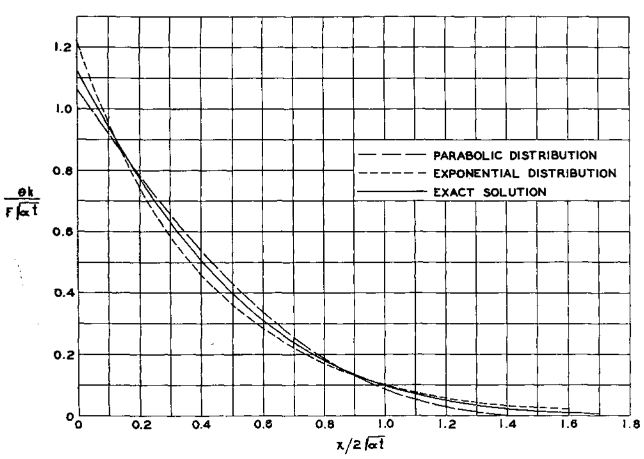

FIG. 2. Internal temperature distribution for different profiles; constant flux to the surface of a semi-infinite solid.

is independent of the factor of proportionality between the surface temperature and time, and is given as

$$q_2^2 = [(147/13)\alpha t]/[(15/13)n + 1]$$

If $n = (1/2)$, which corresponds to a constant flux condition (this result is in reality known a posteriori from the results of the preceding discussion), the penetration depth is

$$q_2^2 = (294/41)\alpha t$$

This result agrees with that obtained by using the flux condition with the variational equation for q_2. Therefore the use of the flux condition together with the variational equation for the penetration depth leads to the penetration depth history given above. This result is the same as that obtained from the variational equation for a prescribed surface temperature history, corresponding to constant flux.

The preceding results have indicated that the variational solutions agree well with the exact solution. In the sections to follow involving semi-infinite solids, the penetration depth will be selected as the independent generalized coordinate whose variational equation will be used in conjunction with the flux condition.

Finite Slab

The remaining portion of this section will present the results for the case of constant heat flux applied to the surface of a finite slab, one surface of which is insulated. The generalized coordinates are indicated in Fig. 1 and the temperature distribution is

$$\theta = (q_1 - q_3)[1 - (x/b)]^2 + q_3 \qquad (23)$$

where x is the coordinate measured from the heated face, b is the thickness of the slab, and q_1 and q_3 are the surface temperatures. In the analysis of finite slabs the q_1 coordinate will be selected as the independent generalized coordinate. The heat flux field, the surface heat flux and the variational equation for q_1 are:

$$\dot{H} = (cb/3)(\dot{q}_i - \dot{q}_3)\left(1 - \frac{x}{b}\right)^3 + cb\dot{q}_3\left(1 - \frac{x}{b}\right)$$

$$\dot{H}_n = (cb/3)(\dot{q}_1 + 2\dot{q}_3)$$

$$10\dot{q}_1 + 32\,\dot{q}_3 = 84(q_1 - q_3)$$

The parameters

$$\varphi_{(\ )} = q_{(\ )}k/Fb$$
$$\tau = \alpha t/b^2$$

where F is the constant heat flux at the surface, will reduce the governing equations to the form:

$$\left.\begin{aligned} 10\,\dot{\varphi}_1 + 32\,\dot{\varphi}_3 &= 84(\varphi_1 - \varphi_3) \\ \dot{\varphi}_1 + 2\dot{\varphi}_3 &= 3 \end{aligned}\right\} \qquad (24)$$

where the second of Eqs. (24) is the flux condition, Eq. (13).

Before this stage of heating begins it is necessary for the penetration depth to equal the slab thickness,

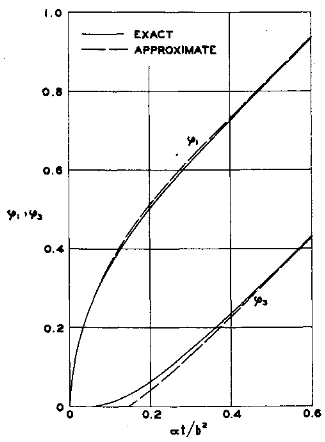

FIG. 3. Surface temperature histories; constant flux to one surface of a finite slab.

and this requires the lapse of a certain time, called the penetration time. After this time the second stage of heating begins which corresponds to the use of the two surface temperature coordinates. The penetration time is evaluated from the results of the semi-infinite case which will be called phase 1 of the heating period. It follows from Table 1 that:

$$(q_1k/Fb) = 1.157(\tau_p)^{1/2}$$

$$q_2/b = 2.59(\tau_p)^{1/2}$$

where $\tau_p = (\alpha/b^2)t_p$ is the nondimensional penetration time. Evaluations yield as the initial conditions for phase 2 of the heating:

$$\tau_p = 0.149; \quad \varphi_1 = 0.445; \quad \varphi_3 = 0 \qquad (25)$$

Typical values of the penetration time for a one inch thick slab are: copper: 0.98 sec; aluminum: 1.73 sec; and stainless steel: 2.12 sec.

Eqs. (24) upon elimination of φ_3 and the use of the initial conditions, Eqs. (25), become

$$\dot{\varphi}_1 + 21\varphi_1 = 21\tau + 7.9888$$

the solution to which is

$$\varphi_1 = \tau + 0.3328 - 0.8374 \exp(-21\tau); \quad \tau > \tau_p$$

The exact solution for the heated face is,[1, 23]

$$\varphi_1 = \tau + (1/3) - (2/\pi^2) \sum_{n=1}^{\infty} (1/n^2) \exp(-n^2\pi^2\tau)$$

If τ is large, the series term contributes a negligible amount to the surface temperature, and the surface temperature becomes linear in time.

Comparison of the variational solution with the exact solution indicated in Ref. 23 is shown in Figs. 3 and 4. These figures show the temperature versus time curves for the surface temperatures, and the internal temperature distribution for times greater and less than the penetration time; agreement is good in all cases.

Heat Flux Proportional to a Power of the Surface Temperature

The case where the boundary heat flux is constant has been considered in the last section together with a discussion of the basic method of approach. In particular, the selection of the independent generalized coordinates and the role of the flux condition were emphasized. This section will treat the case where the flux condition depends upon a power of the surface temperature. The cases to be treated in detail are: (1) Radiation from a semi-infinite solid at a rate proportional to a power of the surface temperature. The initial temperature is θ_0 while the ambient temperature is absolute zero; (2) radiation from a medium at temperature θ_0 to a semi-infinite solid at zero initial temperature; (3) radiation from a finite slab insulated at one face into a medium at zero degrees absolute.

Under (1), the case of Newton's law of cooling and fourth power radiation will be discussed. Newton's law of cooling, which is the linear case, will be used as a check on the variational equations since an exact solution is known.[1]

However, the fourth power radiation case is not tractable by classical methods of solutions due to the nonlinearity of the boundary condition; approximate methods have, however, been used and these will be discussed below.

The first formulation of this fourth power radiation problem is due to Jaeger[24] who used a series method of solution to determine the temperature distribution. The curve for the surface temperature history obtained from a numerical integration of the heat conduction equation was also presented. Abarbanel[25] treated the same problem by a formal method employing the Laplace transform technique. This method reduced the heat conduction equation with the prescribed boundary conditions to an integral equation which was solved numerically.

The problem stated in (2) was investigated by Chambré[26] who followed a similar formulation to that in Ref. 25. The method of solution was a successive approximation scheme from which the surface temperature history was found. In neither of these problems was the internal temperature distribution found, although it can be obtained immediately from the knowledge of the surface temperature.

The heat balance integral method was also used for the case of radiation cooling of a semi-infinite solid,[18]

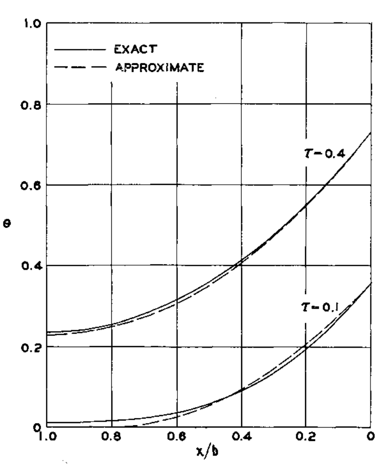

FIG. 4. Internal temperature distribution at different values of time; constant flux to one surface of a finite slab.

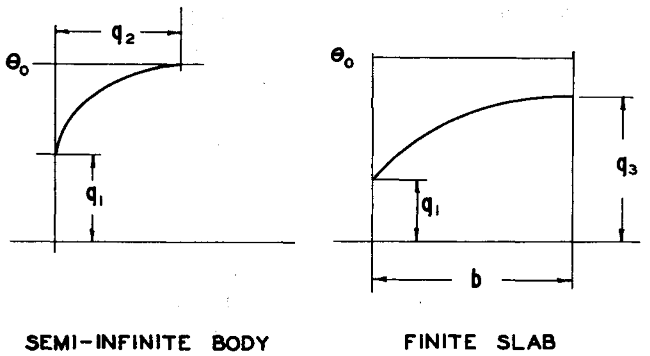

FIG. 5. Temperature profiles in terms of generalized coordinates; heat loss.

together with radiation cooling from a finite slab.[21] This latter case falls under (3) above.

Semi-Infinite Solid

The present method of solution for the case of radiation cooling will employ the generalized coordinates shown in Fig. 5. The temperature distribution is

$$\theta = -(\theta_0 - q_1)[1 - (x/q_2)]^2$$

where θ_0 is the initial temperature.

Introduction of the parameters

$$\psi_1 = 1 - (q_1/\theta_0)$$
$$\eta = q_2 h\, \theta_0^{m-1}/k$$
$$\tau = (h\, \theta_0^{m-1}/k)^2 \alpha t$$

and the applied flux boundary condition

$$\dot{H} = -h\theta_0^m(1 - \psi_1)^m \tag{26}$$

will reduce the flux equation, Eq. (13), and variational equation, Eq. (10), to the form

$$\left.\begin{aligned} \dot{\eta}\psi_1 + \dot{\psi}_1\eta &= 3(1 - \psi_1)^m \\ (\dot{\psi}_1\eta^2/42) + (13/315)\psi_1\eta\dot{\eta} &= (7/30)\psi_1 \end{aligned}\right\} \tag{27}$$

Eqs. (27) are the governing equations for the temperature distribution in a semi-infinite solid which loses heat at a rate proportional to a power m of the surface temperature, where the constant h is the factor of proportionality. If $m = 1$, Eq. (26) corresponds to Newton's law of cooling; if $m = 4$, Eq. (26) corresponds to black-body radiation.

Eqs. (27) are linear in the first derivatives and can be written in the form

$$\left.\begin{aligned} \dot{\eta} &= 147\psi_1 - 45\eta(1 - \psi_1)^m/11\eta\psi_1 \\ \dot{\psi}_1 &= 78\eta(1 - \psi_1)^m - 147\psi_1/11\eta^2 \end{aligned}\right\} \tag{28}$$

which is a convenient form for numerical integration. Thus the heat conduction problem is reduced to an initial value problem under the assumption of a spatial temperature profile. The initial conditions are $\eta = \psi_1 = 0$, corresponding to the initial uniform temperature θ_0. In both cases, $m = 1$ and $m = 4$, the asymptotic solutions to Eqs. (28) for long and short times will be given in addition to the results of the numerical integrations.

If a power series expansion is substituted into either Eqs. (27) or Eqs. (28), the form of the asymptotic solutions can be determined. The short time solutions are:

$$\left.\begin{aligned} (q_1/\theta_0) &= 1 - 1.120\, \tau^{1/2} \\ \eta &= 2.68\, \tau^{1/2},\ \tau \to 0 \end{aligned}\right\} \tag{29}$$

These results agree with those obtained for constant heat flux and they are independent of the power m of the surface temperature. The asymptotic solution for long times does depend however on the power of m and can be written in the form:

$$\left.\begin{aligned} (q_1/\theta_0) &= (0.3141/\tau)^{1/2m} \\ \eta &= 3.36\, \tau^{1/2},\ \tau \to \infty \end{aligned}\right\} \tag{30}$$

The exact solution obtained in Ref. 25 is

$$(q_1/\theta_0) = (1/\pi\tau)^{1/2m},\ \tau \to \infty$$

Good agreement exists, therefore, between the asymptotic solutions.

The exact solution for the surface temperature for the case $m = 1$ is[1]

$$(q_1/\theta_0) = e^{\tau}\, \mathrm{erfc}(\tau^{1/2})$$

from which the short and long time solutions can be obtained:

$$(q_1/\theta_0) = 1 - 2(\tau/\pi)^{1/2},\ \tau \to 0$$
$$(q_1/\theta_0) = 1/(\pi\tau)^{1/2},\quad \tau \to \infty$$

The asymptotic solutions to Eqs. (28) for the case $m = 1$ are given by Eqs. (29) and, from Eqs. (30), for $\tau \to \infty$,

$$(q_1/\theta_0) = 0.560/\tau^{1/2}$$
$$\eta = 3.36\tau^{1/2}$$

The asymptotic results for the case $m = 4$ are outlined in Table 2, together with the results obtained by Abarbanel,[25] Schneider,[21] and Goodman.[18]

Finally, Figs. 6 to 9 present the surface temperature and penetration depth time histories together with the internal temperature distributions. These results were obtained by integrating numerically Eqs. (28) by the Runge-Kutta technique; the integration was carried out to $\tau = 1$. The results for $m = 1$ are compared with the known exact solution,[1] and the results for $m = 4$ are compared with those of Abarbanel.[25] The

TABLE 2. Values of Asymptotic Solutions for the Case $m = 4$

	Short Time Solution	Long Time Solution
Variational Solution	$\psi = 1.120\ \tau^{1/2}$	$1 - \psi = 0.865/\tau^{1/8}$
	$\eta = 2.68\ \tau^{1/2}$	$\eta = 3.36\ \tau^{1/2}$
Heat Balance Integral Parabolic	$\psi = 1.225\ \tau^{1/2}$	$1 - \psi = 0.872/\tau^{1/8}$
	$\eta = 2.45\ \tau^{1/2}$	$\eta = 3.46\ \tau^{1/2}$
Cubic	$\psi = 1.153\ \tau^{1/2}$	$1 - \psi = 0.884/\tau^{1/8}$
	$\eta = 3.46\ \tau^{1/2}$	$\eta = 4.90\ \tau^{1/2}$
Exact	$\psi = 1.128\ \tau^{1/2}$	$1 - \psi = 0.867/\tau^{1/8}$

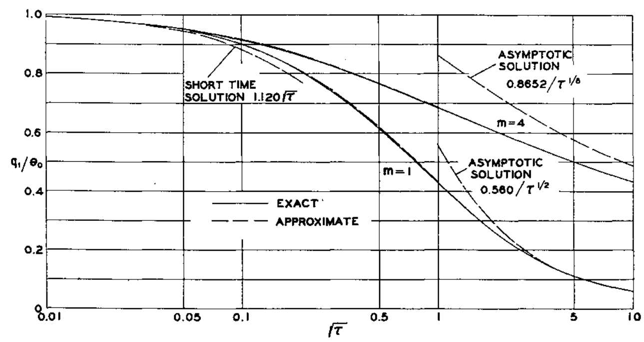

FIG. 6. Surface temperature history; heat loss from the surface of a semi-infinite solid at a rate proportional to a power m of the surface temperature.

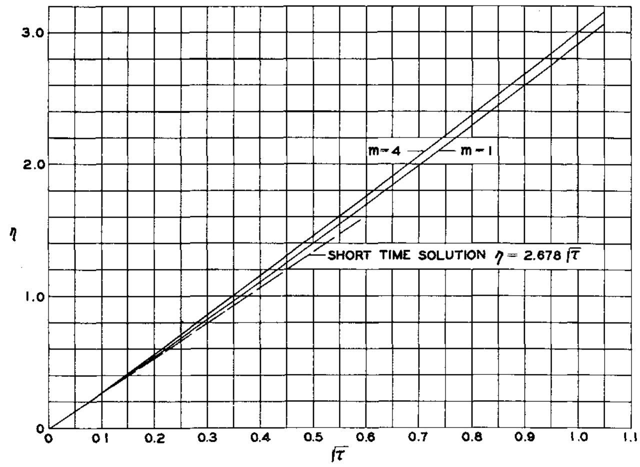

FIG. 7. Penetration depth history; heat loss from the surface of a semi-infinite solid at a rate proportional to a power m of the surface temperature.

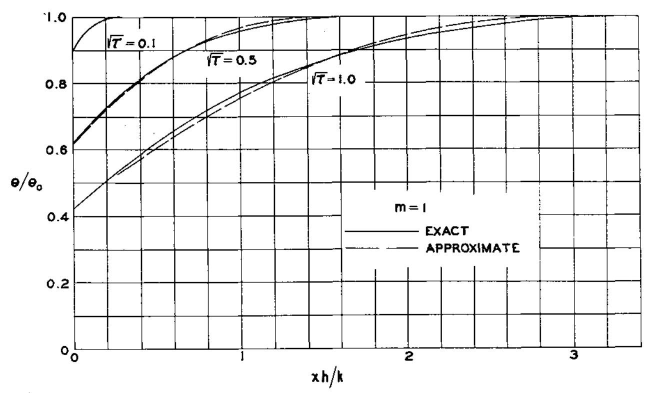

FIG. 8. Internal temperature distribution at different values of time; heat loss from the surface of a semi-infinite solid at a rate proportional to the first power of the surface temperature.

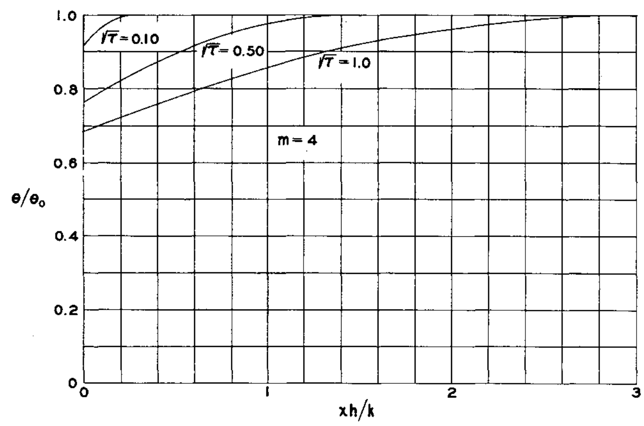

FIG. 9. Internal temperature distribution at different values of time; heat loss from the surface of a semi-infinite solid at a rate proportional to the fourth power of the surface temperature.

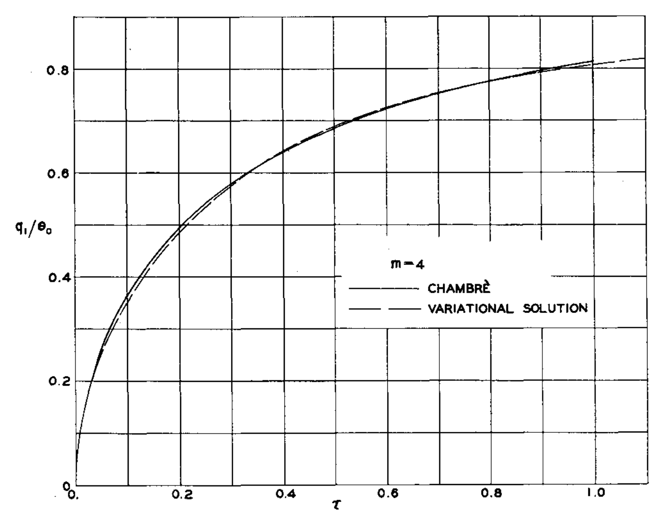

FIG. 10. Surface temperature history; heat input to the surface of a semi-infinite solid at a rate proportional to the fourth power of the surface temperature.

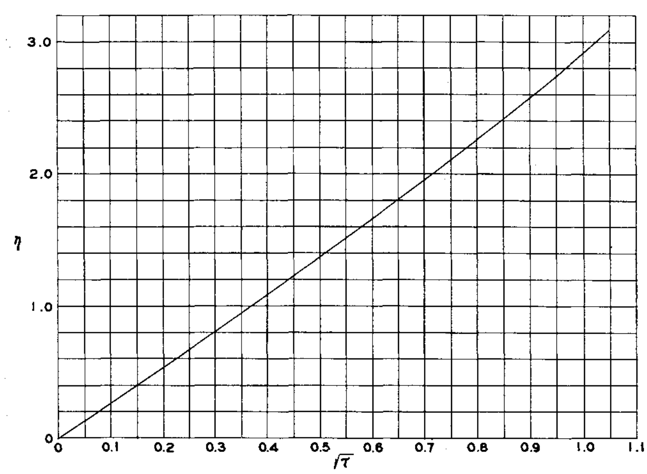

FIG. 11. Penetration depth history; heat input to the surface of a semi-infinite solid at a rate proportional to the fourth power of the surface temperature.

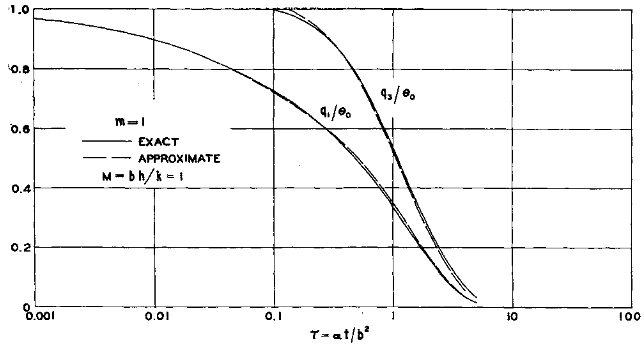

FIG. 12. Surface temperature histories; heat loss from a finite slab at a rate proportional to the first power of the surface temperature.

agreement is good in both cases and over part of the curve no distinction between results is evident.

The case of heat input to a semi-infinite solid at a rate proportional to the fourth power of the surface temperature was treated by Chambré[26] and will also be considered here. In this example the generalized coordinates are indicated in Fig. 1; the variational equation is Eq. (20) and the flux condition is

$$\dot{\psi}\eta + \dot{\eta}\psi = 3(1 - \psi^4)$$

In this case the equations can again be integrated numerically to obtain the results for the surface temperature and penetration depth time histories shown in Figs. 10 and 11. The short time solutions are given by

$$\psi = 1.120\tau^{1/2}$$
$$\eta = 2.68\tau^{1/2}$$

The previous example completes the discussion of the cases (1) and (2) mentioned at the beginning of this section; the subsequent discussion will treat finite slabs. Of course, the results of phase 1 corresponding to the semi-infinite body are used as the initial values for phase 2.

Finite Slab

Fig. 5 indicates the generalized coordinates to be used in this case for a finite slab. The slab is of thickness b and insulated at the inner face; the temperature distribution is

$$\theta/\theta_0 = (\varphi_3 - \varphi_1)\xi^2 - \varphi_3$$

where

$$\varphi_{(\ )} = 1 - (q_{(\ )}/\theta_0)$$
$$\xi = 1 - (x/b)$$

The calculation of the heat flow field and the determination of the variational equation follows the same outline as before; it will not therefore be repeated. The results are

$$\left.\begin{aligned} 10\dot{\varphi}_1 + 32\dot{\varphi}_3 &= 84(\varphi_1 - \varphi_3) \\ \dot{\varphi}_1 + 2\dot{\varphi}_3 &= 3M(1 - \varphi_1)^m \end{aligned}\right\} \quad (31)$$

where

$$\tau = \alpha t/b^2, \quad M = bh\theta_0^{m-1}/k$$

Eqs. (31) are the governing equations for the coordinates q_1 and q_3. The first equation is the variational equation and the second equation is the flux condition.

The cases considered are again $m = 1$ and $m = 4$ with $M = 1$; this latter value determines a slab thickness for a given heating rate. The penetration values obtained from the semi-infinite slab analysis for both cases are

$$m = 1: \quad \tau_p = 0.1296, \quad \varphi_1 = 0.305, \quad \varphi_3 = 0 \qquad m = 4: \quad \tau_p = 0.1225, \quad \varphi_1 = 0.190, \quad \varphi_3 = 0$$

These values are the prescribed initial conditions for phase 2. Eqs. (31) can be rewritten in the form

$$\left.\begin{aligned} \dot{\varphi}_1 &= 8(1 - \varphi_1)^m - 14(\varphi_1 - \varphi_3) \\ \dot{\varphi}_3 &= 7(\varphi_1 - \varphi_3) - 2.5(1 - \varphi_1)^m \end{aligned}\right\} \quad (32)$$

which is again in a form convenient for numerical integration.

In the case $m = 1$, the equations are linear and the solution is obtained immediately. The details of the solution are given in Ref. 29 and the complete time history of the surface temperatures is shown in Fig. 12. The agreement between the approximate and exact solution is good.

The case $m = 4$ has not been solved exactly, so that no comparison with an exact solution is possible. However the heat balance integral method has led to an approximate solution which will be used for comparison with the variational solution.[21] Eqs. (32) are nonlinear and must be integrated numerically; in this case, as well as all cases in the present paper, the numerical integration is performed by the Runge-Kutta technique. Fig. 13 shows the results of the numerical calculations and the asymptotic solution for long times.

This asymptotic solution for the general case obtained from Eqs. (32) is

$$\varphi_1 = \varphi_3 = 1 - (1/(m - 1)M\tau)^{1/m-1}, \ m > 1 \quad (33)$$

which for $m = 4$ becomes

$$\varphi_1 = 1 - (1/3\ M\tau)^{1/3}$$

This result is to be expected physically since for long

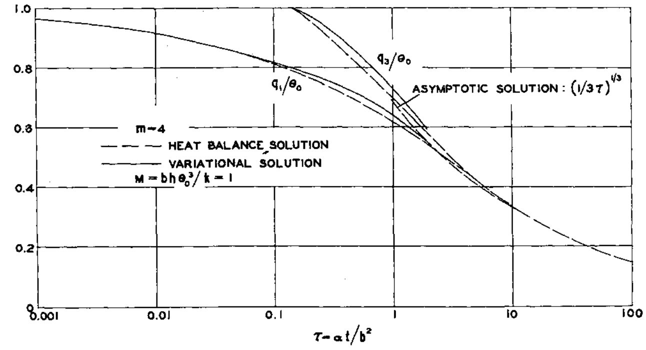

FIG. 13. Surface temperature histories; heat loss from a finite slab at a rate proportional to the fourth power of the surface temperature.

times the slab may be considered a highly conducting sheet of infinite conductivity whose rate of change of temperature equals the heat loss at the surface. The expression for this is

$$\dot{\varphi}_1 = -M\varphi_1{}^m$$

Integration yields

$$\varphi_1 = (1/M(m-1)\tau + C)^{1/m-1}$$

where C is a constant of integration; this equation agrees with Eq. (33) for large values of τ.

It is clear that the present method can also be applied to problems involving powers of the surface temperature corresponding to convection and combined heat inputs. An additional advantage of the present method, among others mentioned previously, is that the problem is reduced to an initial value problem from which important information in the form of asymptotic solutions can be obtained without performing the numerical integration

Variable Thermal Properties

In the case of variable thermal properties the variational formulation is modified as outlined by Biot.[5] The example to be treated is the determination of the temperature distribution in a semi-infinite solid whose surface undergoes a stepwise variation in temperature. The thermal conductivity is of the form

$$k = k_0[1 + \bar{\alpha}(\theta/\theta_s)]$$

where k_0 is the value of the thermal conductivity at $\theta = 0$ and $\bar{\alpha}$ is a prescribed constant; the specific heat, in addition, is assumed constant. This is a simplified example and a more detailed case for constant heat input is treated in Refs. 29 and 31.

The above example was first treated by Yang,[27] and the temperature distribution was obtained numerically by a method of successive approximations. This method of solution is not readily adaptable to other physical problems. In a later paper, Yang and Szewczyk,[28] employed the heat balance integral method of Goodman for the same problem outlined above. However, it was pointed out that the method using polynomial profiles leads to physically inconsistent results when there is strong dependence of thermal properties on temperature. For this reason an exponential type of profile was used.

Goodman,[20] considered the same problem using both a cubic and quartic temperature profile. In addition, the importance of imposing correct constraint conditions, that is, satisfying the heat conduction equation at a specified number of points, was stressed. It was shown that an improper choice of constraint conditions will lead to poor agreement with the exact solution, and that additional difficulties will arise for certain ranges of material parameters. Goodman overcame these difficulties by changing the constraint conditions.

In the use of the variational principle, the assumption of constant heat capacity reduces the thermal potential to its usual form[5]; this, together with the parabolic temperature distribution

$$\theta = \theta_s[1 - (x/q_2)]^2$$

leads to the following expressions for the derivative of the thermal potential and the thermal force:

$$\partial V/\partial q_2 = (1/10)\, c\theta_s{}^2$$

$$Q = (1/3)\, c\theta_s{}^2$$

The expression

$$\frac{\partial D}{\partial \dot{q}_2} = \frac{\theta_s{}^2 c\, q_2\dot{q}_2}{9\alpha_0}\int_0^1 \frac{(4\zeta^6 - 12\zeta^5 + 9\zeta^4)\, d\zeta}{1 + \bar{\alpha}\zeta^2}$$

for the derivative of the dissipation function is obtained upon evaluation of the heat flux field from the assumed temperature distribution. The parameter α_0 is the value of the diffusivity at $\theta = 0$ and

$$\zeta = 1 - (x/q_2)$$

The basic differential equation, Eq. (10), for the penetration depth admits the solution

$$q_2 = A(\alpha t)^{1/2}$$

where the value of A depends upon the value of the thermal conductivity parameter $\bar{\alpha}$. The results obtained for three values of $\bar{\alpha}$ are:

$\bar{\alpha}$	A
1/2	3.83
0	3.36
−1/2	2.76

The result for $\bar{\alpha} = 0$ was also obtained in Ref. 5. In the present problem no need exists for the flux condition employed in the previous examples since the boundary condition is one of prescribed surface temperature.

The temperature distribution is obtained immediately, once the value of the penetration depth is determined as a function of time. The results for the internal temperature distribution as a function of $\bar{\alpha}$ together with the exact solution obtained numerically by Yang,[27] are shown in Fig. 14. These results are in better agreement with the exact solution than those obtained by the heat balance integral method.[20]

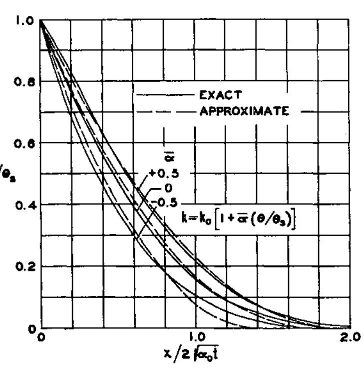

FIG. 14. Internal temperature distribution; step-wise variation in surface temperature, thermal conductivity a linear function of temperature.

Concluding Remarks

Biot's variational principle has been applied to a number of different one-dimensional heat conduction problems. These problems have shown the applicability of the variational principle to problems involving prescribed heat flux boundary conditions and to those with temperature-dependent material properties.

A method was introduced for including boundary conditions when these are expressed as prescribed heat fluxes. The idea behind this was overall energy balance within the body, which is a constraint condition to be satisfied by the time histories of the generalized coordinates.

The variational principle was then applied to the well-known problem of constant surface heat flux in order to present the technique and provide a basis for the remaining sections. The equivalence of the result obtained in applying the variational principle for a prescribed surface temperature history to that obtained for a prescribed heat flux was also pointed out. Radiation cooling due to fourth power radiation from semi-infinite solids and finite slabs together with radiation according to Newton's law of cooling was then treated. Finally, the introduction of temperature-dependent material properties was discussed and the determination of the temperature distribution in a semi-infinite solid was investigated.

The results have been presented in graphical form and comparisons have been made with other approximate solutions whenever they are available. Agreement in all cases was found to be good. In addition to the accuracy, which is most important in any approximate solution, the variational method has been shown for one-dimensional problems to provide a systematic means of deducing the temperature history.

Additional applications of the variational principle to problems with time-dependent boundary conditions and to the problem of the heating of a semi-infinite solid whose properties are temperature-dependent can be found in Refs. 29, 30, and 31. This paper is a summary of some of the results to be found in those references.

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